Exam - Statistics (WBMA009-05) 2021/2022

Date and time: November 10, 2021, 18.45-21.45h
Place: Exam Hall 2, Blauwborgje 4

Rules to follow:

- This is a closed book exam. Consultation of books and notes is **not** permitted. You can use a simple (non-programmable) calculator.
- Write your name and student number onto each paper sheet. There are 4 exercises and you can reach 90 points. ALWAYS include the relevant equation(s) and/or short descriptions.
- **We wish you success with the completion of the exam!**

START OF EXAM

1. **Asymptotic confidence intervals and tests.** **25**
Consider a random sample $X_1, \dots, X_n$ from a negative Binomial distribution with known parameter $r \in \mathbb{N}$ and unknown probability parameter $\theta \in (0,1)$. Recall that the density and the expectation are

$$f(x) = \binom{x+r-1}{x} \cdot (1-\theta)^r \cdot \theta^x \quad (x \in \mathbb{N}_0), \qquad E[X] = \frac{\theta r}{1-\theta}$$

(a) Show that the ML estimator of θ is: $\hat{\theta}_{ML} = \bar{X}/(r+\bar{X})$.
Check via the 2nd derivative if this is really a maximum point. 5

(b) Show that the expected Fisher information (for a sample size 1) is

$$I(\theta) = \frac{r}{\theta \cdot (1-\theta)^2} \qquad \boxed{5}$$

From now on we assume that $r = 2$, $n = 20$ and that $\bar{X} = 8$ has been observed. And we use the quantiles provided in Table 1 on page 2.

(c) Make use of the asymptotic normality of the ML estimator and give a two-sided asymptotic 95% confidence interval $[L, U]$ for θ. 5

(d) Make use of the asymptotic normality of the ML estimator and give a one-sided asymptotic 95% confidence interval $(-\infty, U]$ for θ. 5

(e) Check whether a score-test to the level $\alpha = 0.02$ would reject the null hypothesis $H_0 : \theta = 0.9$ in favour of the alternative $H_1 : \theta \neq 0.9$. 5
HINT: Score test: $\frac{d}{d\theta} l_X(\theta)/\sqrt{n \cdot I(\theta)}$ is asymptotically $N(0,1)$ distributed.

α	0.5	0.75	0.9	0.95	0.975	0.99	0.99997
q_α	0	0.7	1.3	1.6	2	2.3	4

Table 1: Approximate quantiles q_α of the $\mathcal{N}(0,1)$ distribution.

2. **Random sample.** $\boxed{\mathbf{30}}$
   Consider a random sample
   $$X_1, \ldots, X_n \sim \mathcal{F}_\theta$$
   from a distribution that depends on a parameter $\theta > 0$ and whose density is:
   $$f_\theta(x) = \frac{1}{2} \cdot \theta^3 \cdot x^2 \exp\{-x\theta\} \qquad (x > 0)$$
   (a) Give a sufficient statistic for θ. $\boxed{5}$
   (b) Compute the ML estimator of θ. $\boxed{5}$
   HINT: Check via the 2nd derivative whether you have a global maximum.
   (c) Compute the Fisher information $I(\theta)$ for a sample of size $n = 1$. $\boxed{5}$
   (d) Assume that $n = 81$ and $\hat{\theta}_{ML} = 3$. Give an asymptotic one-sided 95% confidence interval (of the type $[L, \infty]$) for θ. $\boxed{5}$
   HINTS:
   You can assume that all regularity conditions are fulfilled.
   See Table 1 for the relevant quantiles.
   (e) Consider the simple test problem
   $$H_0 : \theta = 4 \ \textbf{vs.} \ H_1 : \theta = 2$$
   Show that a statistical test that rejects H_0 if $\sum_{i=1}^{n} X_i > k_0$, where $k_0 > 0$ is a constant, is the UMP test for this test problem. $\boxed{10}$

3. **Test level, power and p-value of a statistical test.** $\boxed{\mathbf{20}}$
   Consider a random sample of size n from a Gaussian distribution with known variance parameter $\sigma^2 = 4$:
   $$X_1, \ldots, X_n \sim \mathcal{N}(\mu, 4)$$
   and the simple test problem
   $$H_0 : \mu = 0 \ \textbf{vs.} \ H_1 : \mu = -0.54$$
   The null hypothesis is rejected when $\bar{x}_n \leq -0.4$.
   For solving the exercise use <u>and only use</u> the quantiles provided in Table 1.
   (a) Given sample size $n = 100$, what is the test level? $\boxed{5}$
   (b) Given sample size $n = 100$, what is the power of the test? $\boxed{5}$
   (c) Given sample size $n = 100$, assume that $\bar{x}_n = -0.14$ was observed.
   What is the p-value of the test? $\boxed{5}$
   (d) How large must n (at least) be, so that the test has power 0.9? $\boxed{5}$

4. **Sample from Poisson distribution.** $\boxed{\mathbf{15}}$
Let $X_1, \ldots, X_n$ be a sample from a Poisson distribution with density:

$$p(x|\lambda) = e^{-\lambda} \cdot \frac{\lambda^x}{x!} \qquad (x \in \mathbb{N}_0)$$

where $\lambda > 0$ is an unknown parameter.

Recall that $T(X_1, \ldots, X_n) := \sum_{i=1}^{n} X_i$ has a Poisson distribution with parameter $n\lambda$.

(a) Show that $T(X_1, \ldots, X_n) := \sum_{i=1}^{n} X_i$ is a sufficient statistic for λ. $\boxed{5}$

(b) Derive the uniform most powerful (UMP) test for the test problem

$$H_0 : \lambda \leq \lambda_0 \quad \text{vs.} \quad H_1 : \lambda > \lambda_0$$

to the significance level $\alpha = 0.05$. $\boxed{10}$
HINT: In your solutions you can use the symbol $q_{\lambda,\alpha}$ to denote the α quantile of a Poisson distribution with parameter λ.

Solutions Exercise 1

1(a): Compute the log likelihood:

$$\begin{aligned} l_X(\theta) &= \log\left(\prod_{i=1}^n \binom{x_i+r-1}{x_i} \cdot (1-\theta)^r \cdot \theta^{x_i}\right) \\ &= \log\left((\prod_{i=1}^n \binom{x_i+r-1}{x_i}) \cdot (1-\theta)^{nr} \cdot \theta^{\sum_{i=1}^n x_i}\right) \\ &= \log\left(\prod_{i=1}^n \binom{x_i+r-1}{x_i}\right) + nr\log(1-\theta) + (\sum_{i=1}^n x_i)\log(\theta) \end{aligned}$$

Take the derivative w.r.t. θ and set it to 0:

$$\frac{-nr}{1-\theta} + \frac{\sum_{i=1}^n x_i}{\theta} = 0 \;\Leftrightarrow\; -nr\theta + (\sum_{i=1}^n x_i)(1-\theta) = 0 \Leftrightarrow -(nr + \sum_{i=1}^n x_i)\theta + \sum_{i=1}^n x_i = 0$$
$$\Leftrightarrow\; \theta = \frac{\sum_{i=1}^n x_i}{nr + \sum_{i=1}^n x_i} \Leftrightarrow \theta = \frac{\bar{x}}{r+\bar{x}}$$

For the second order derivative we have:

$$\frac{d^2}{d\theta^2} l_X(\theta) = \frac{-nr}{(1-\theta)^2} - \frac{\sum_{i=1}^n x_i}{\theta^2} < 0 \qquad (0 < \theta < 1)$$

This confirms that $\hat{\theta}_{ML} = \bar{X}/(r+\bar{X})$ globally maximizes the (log-)likelihood.

1(b): For $n = 1$ we have the 2nd order derivative of the log likelihood:

$$\frac{d^2}{d\theta^2} l_{X_1}(\theta) = \frac{-r}{(1-\theta)^2} - \frac{X_1}{\theta^2}$$

Compute the Fisher Information:

$$\begin{aligned} I(\theta) &= -E_\theta\left[\frac{d^2}{d\theta^2} l_{X_1}(\theta)\right] = E_\theta\left[\frac{r}{(1-\theta)^2} + \frac{X_1}{\theta^2}\right] = \frac{r}{(1-\theta)^2} + \frac{E[X_1]}{\theta^2} \\ &= \frac{r}{(1-\theta)^2} + \frac{\frac{r\theta}{(1-\theta)}}{\theta^2} = \frac{r}{(1-\theta)^2} + \frac{r\theta}{(1-\theta)\theta^2} = \frac{r\theta + r(1-\theta)}{(1-\theta)^2\theta} = \frac{r}{\theta(1-\theta)^2} \end{aligned}$$

1(c): Asymptotically $\sqrt{I(\theta)}\sqrt{n} \cdot (\hat{\theta}_{ML} - \theta) \sim \mathcal{N}(0,1)$, hence:

$$\begin{aligned} & P(q_{0.025} \le \sqrt{I(\theta)}\sqrt{n} \cdot (\hat{\theta}_{ML} - \theta) \le q_{0.975}) = 0.95 \\ \Leftrightarrow\; & P(\hat{\theta}_{ML} - \frac{q_{0.975}}{\sqrt{I(\theta)} \cdot \sqrt{n}} \le \theta \le \hat{\theta}_{ML} - \frac{q_{0.025}}{\sqrt{I(\theta)} \cdot \sqrt{n}}) = 0.95 \end{aligned}$$

With $q_{0.975} = 2$ and $q_{0.025} = -2$, and $I(\theta)$ being replaced by $I(\hat{\theta}_{ML})$, we get the CI:

$$\hat{\theta}_{ML} \pm 2/(\sqrt{I(\hat{\theta}_{ML})} \cdot \sqrt{n})$$

Here we have $\hat{\theta}_{ML} = 0.8$ and $2/(\sqrt{I(\hat{\theta}_{ML})}\sqrt{n}) = 2/(\sqrt{2/(0.8 \cdot 0.2^2)}\sqrt{20} \approx 0.057$.

So the two-sided CI is: $[0.74, 0.86]$.

1(d): Like part (c), but here we use:

$$P(q_{0.05} \leq \sqrt{I(\theta)}\sqrt{n} \cdot (\hat{\theta}_{ML} - \theta)) = 0.95 \Leftrightarrow P(\hat{\theta}_{ML} - \frac{q_{0.05}}{\sqrt{I(\theta)} \cdot \sqrt{n}} \geq \theta) = 0.95$$

With $q_{0.05} = -1.6$ the one-sided 95% CI for θ is: $(-\infty, \hat{\theta}_{ML} + \frac{1.6}{\sqrt{I(\hat{\theta}_{ML})} \cdot \sqrt{n}}]$
Here we have $\hat{\theta}_{ML} = 0.8$ and $\frac{1.6}{\sqrt{I(\hat{\theta}_{ML})}\sqrt{n}} = \frac{1.6}{\sqrt{\frac{2}{0.8 \cdot 0.2^2}}\sqrt{20}} \approx 0.045$.

So the one-sided CI is: $(-\infty, 0.845]$.

1(e): Asymptotically: $\frac{\frac{d}{d\theta} l_X(\theta)}{\sqrt{n \cdot I(\theta)}} \sim N(0,1)$ where $\frac{d}{d\theta} l_X(\theta) = \frac{-nr}{1-\theta} + \frac{\sum_{i=1}^n x_i}{\theta}$.

Given $r = 2$, $\bar{X} = 8$ and $n = 20$ and $\theta_0 = 0.9$ we get:
$\frac{-nr}{1-\theta} + \frac{\sum_{i=1}^n x_i}{\theta} = \frac{-40}{1-0.9} + \frac{20 \cdot 8}{0.9} \approx -222$ and $\sqrt{n \cdot I(\theta)} = \sqrt{20 \cdot \frac{2}{0.9 \cdot 0.1^2}} \approx 66.67$

Therefore the score test statistic takes the value: $\frac{\frac{d}{d\theta} l_X(\theta)}{\sqrt{n \cdot I(\theta)}} = \frac{-222}{66.67} \approx -3.33$. As the value is lower than the $q_{0.01}$ quantile -2.3 of the $N(0,1)$, **the score test would reject the null hypothesis** to the level 0.02.

Solutions Exercise 2

2(a): We have the likelihood

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{2} \cdot \theta^3 \cdot x_i^2 \cdot \exp\{-x_i\theta\} = \frac{1}{2^n} \cdot \theta^{3n} \cdot \left(\prod_{i=1}^{n} x_i^2\right) \cdot \exp\{-\theta \sum_{i=1}^{n} x_i\}$$

And we can factorize into:

$$L(\theta) = g(x_1, \dots, x_n) \cdot h(\sum_{i=1}^{n} x_i, \theta)$$

where

$$g(x_1, \dots, x_n) := \frac{1}{2^n} \cdot \prod_{i=1}^{n} x_i^2 \quad \text{and} \quad h(\sum_{i=1}^{n} x_i, \theta) := \theta^{3n} \cdot \exp\{-\theta \sum_{i=1}^{n} x_i\}$$

2(b): From the likelihood we get the log-likelihood:

$$l(\theta) = -n\log(2) + 3n\log(\theta) + 2\sum_{i=1}^{n} \log(x_i) - \theta \sum_{i=1}^{n} x_i$$

We take the derivative w.r.t. θ and we set it to zero:

$$l'(\theta) = \frac{3n}{\theta} - \sum_{i=1}^{n} x_i = 0 \Leftrightarrow \theta = \frac{3n}{\sum_{i=1}^{n} x_i} = \frac{3}{\bar{x}}$$

We compute the 2nd derivative:

$$l''(\theta) = -\frac{3n}{\theta^2}$$

As the 2nd derivative is always negative, we indeed have a maximum point. Hence

$$\hat{\theta}_{ML} = \frac{3}{\bar{x}}$$

2(c): We compute the Fisher information (with $n = 1$)

$$I(\theta) = -E[l''(\theta)] = -E\left[-\frac{3}{\theta^2}\right] = \frac{3}{\theta^2}$$

2(d): For large n we have:

$$\sqrt{n} \cdot (\hat{\theta}_{ML} - \theta) \sim \mathcal{N}\left(0, \frac{1}{I(\theta)}\right)$$

Plugging in $n = 81$ and replacing $I(\theta) = \frac{3}{\theta^2}$ by the observed Fisher information

$$I(\hat{\theta}_{ML}) = \frac{3}{3^2} = \frac{1}{3}$$

we get

$$9 \cdot (\hat{\theta}_{ML} - \theta) \sim \mathcal{N}(0,3) \Leftrightarrow \frac{9}{\sqrt{3}} \cdot (\hat{\theta}_{ML} - \theta) \sim \mathcal{N}(0,1)$$

so that

$$P(\frac{9}{\sqrt{3}} \cdot (\hat{\theta}_{ML} - \theta) < q_{0.95}) = 0.95$$

Solving for θ yields:

$$P(\theta > \hat{\theta}_{ML} - q_{0.95} \cdot \frac{\sqrt{3}}{9}) = 0.95$$

With $\hat{\theta}_{ML} = 3$ and $q_{0.95} = 1.6$ (see Table 1) we have the one-sided 95% confidence interval:

$$[2.69; \infty]$$

2(e): The UMP test rejects H_0 if the likelihood ratio $W(X)$ is smaller than a constant k. We have:

$$W(X) = \frac{L(4)}{L(2)} = \frac{\frac{1}{2^n} \cdot 4^{3n} \cdot \left(\prod_{i=1}^{n} X_i^2\right) \cdot \exp\{-4\sum_{i=1}^{n} X_i\}}{\frac{1}{2^n} \cdot 2^{3n} \cdot \left(\prod_{i=1}^{n} X_i^2\right) \cdot \exp\{-2\sum_{i=1}^{n} X_i\}} = 2^{3n} \cdot \exp\{-2\sum_{i=1}^{n} X_i\}$$

$W(X)$ is a monotone decreasing function in $\sum_{i=1}^{n} X_i$, so that we have the equivalence:

$$W(X) < k \Leftrightarrow \sum_{i=1}^{n} X_i > k_0$$

This shows that a test who rejects H_0 if $\sum_{i=1}^{n} X_i > k_0$ is UMP.

Solutions Exercise 3

3(a): Under H_0 we have

$$\sqrt{n} \cdot \frac{(\bar{X}_n - 0)}{2} \sim \mathcal{N}(0,1)$$

For $n = 100$ this means: $5 \cdot \bar{X}_{100} \sim \mathcal{N}(0,1)$ and

$$\bar{X}_{100} < -0.4 \Leftrightarrow 5 \cdot \bar{X}_{100} < -2$$

From Table 1 we see that -2 corresponds to the $q_{0.025}$ quantile, so that the test is (at least) to the level 0.025.

3(b): Under H_1 we have

$$5 \cdot (\bar{X}_{100} + 0.54) \sim \mathcal{N}(0,1)$$

$$\bar{X}_{100} < -0.4 \Leftrightarrow \bar{X}_{100} + 0.54 < 0.14 \Leftrightarrow 5(\bar{X}_{100} + 0.54) < 0.7$$

From Table 1 we see that 0.7 corresponds to the $q_{0.75}$ quantile, so that the power of the test is 0.75.

3(c): Under H_0 we have

$$5 \cdot (\bar{X}_{100} - 0) \sim \mathcal{N}(0,1)$$

We reject H_0 if $\bar{X}_{100}$ takes values lower than a threshold. Hence, the p-value **p** must fulfill:

$$5\bar{X}_{100} = q_{\mathbf{p}} \Leftrightarrow -0.7 = q_{\mathbf{p}}$$

From $q_{0.75} = 0.7$ it follows $q_{0.25} = -0.7$, so that $\mathbf{p} = 0.25$

3(d): Under H_1 we have

$$\sqrt{n} \cdot \frac{(\bar{x}_n + 0.54)}{2} \sim \mathcal{N}(0,1)$$

and the relationship:

$$\bar{x}_n < -0.4 \Leftrightarrow \sqrt{n} \cdot \frac{(\bar{x}_n + 0.54)}{2} < 0.07 \cdot \sqrt{n}$$

From Table 1 we see that the 0.9 quantile corresponds to $q_{0.9} = 1.3$.
To reach a power of 0.9 we thus need:

$$0.07 \cdot \sqrt{n} = q_{0.9} = 1.3 \Leftrightarrow n = \left(\frac{1.3}{0.07}\right)^2 \approx 344.9$$

That is, the sample size must be at least $n = 345$.

Solutions Exercise 4

4(a) The joint density is:

$$p(x_1,\dots,x_n|\lambda)=\prod_{i=1}^{n}p(x_i|\lambda)=e^{-n\lambda}\cdot\frac{\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}=h\left(\sum_{i=1}^{n}x_i|\lambda\right)\cdot g(x_1,\dots,x_n)$$

where

$$h\left(\sum_{i=1}^{n}x_i|\lambda\right)=e^{-n\lambda}\cdot\lambda^{\sum_{i=1}^{n}x_i}\quad\text{and}\quad g(x_1,\dots,x_n)=\frac{1}{\prod_{i=1}^{n}x_i!}$$

It follows (factorization theorem) that $T(X_1,\dots,X_n)=\sum_{i=1}^{n}X_i$ is sufficient statistic.

4(b) Let $\lambda_1>\lambda_0$ and compute the joint density ratio:

$$\begin{aligned}W(X_1,\dots,X_n)&=\frac{p(X_1,\dots,X_n|\lambda_0)}{p(X_1,\dots,X_n|\lambda_1)}\\&=\frac{e^{-n\lambda_0}\cdot\frac{\lambda_0^{(\sum_{i=1}^{n}X_i)}}{\prod_{i=1}^{n}X_i!}}{e^{-n\lambda_1}\cdot\frac{\lambda_1^{(\sum_{i=1}^{n}X_i)}}{\prod_{i=1}^{n}X_i!}}=e^{n(\lambda_1-\lambda_0)}\cdot\left(\frac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^{n}X_i}\end{aligned}$$

Because of $\lambda_1>\lambda_0$ the density ratio is a monotone decreasing function in the sufficient statistic $\sum_{i=1}^{n}X_i$.

We reject H_0 if

$$e^{n(\lambda_1-\lambda_0)}\cdot\left(\frac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^{n}X_i}<k\quad\Leftrightarrow\quad\sum_{i=1}^{n}X_i>k_0$$

Under H_0 the statistic $\sum_{i=1}^{n}X_i$ has a Poisson distribution with parameter $n\lambda_0$. Therefore, the decision rule is to reject the null hypothesis H_0 when $\sum_{i=1}^{n}X_i$ takes a value equal to or larger than $q_{n\lambda_0,\,0.95}$ quantile.